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Critical-level behaviour and wave amplification of a gravity wave incident upon a shear layer

By I. A. ELTAYEB AND J. F. McKENZIE

Department of Mathematics, University of Khartoum, Sudan

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The properties of reflexion, refraction and absorption of a gravity wave incident upon a shear layer are investigated. It is shown that one must expect these properties to be very different depending upon the parameters (such as the Richardson number Ri, the wavelength normalized by the length scale of the shear and the ratio of the flow speed to the phase speed of the wave) characterizing the interaction of a gravity wave with a shear layer. In particular, it is shown that for all Richardson numbers there is a discontinuity in the *net* wave-action flux across the critical level, i.e. at a height where the flow speed matches the horizontal phase speed of the wave. When $Ri > \frac{1}{4}$, this is accompanied by absorption of part of the energy of the incident wave into the mean flow. In addition it is shown that the phenomenon of wave amplification (over-reflexion) can arise provided that the ultimate shear flow speed exceeds the horizontal phase speed of the wave and Ri is less than a certain critical value $Ri_c \simeq 0{\cdot}1129$, in which case the reflected wave extracts energy from the streaming motion. It is also pointed out that wave amplification can lead to instability if the boundary conditions are altered in such a way that the system can behave like an 'amplifier'.

1. Introduction

In this paper we investigate the reflexion and refraction of gravity waves by a shear layer. It is well known (see, for example, Booker & Bretherton 1967) that, if the component in the direction of the streaming of the phase speed of a gravity wave matches the flow speed at some point, a critical layer develops and most of the energy of the wave is absorbed into the mean flow for Richardson numbers of order one or more. On the other hand, an analysis of the reflexion and stability of gravity waves in the presence of a shear layer by Jones (1968) has revealed that wave amplification (or over-reflexion) may occur. Following this, McKenzie (1972) showed that gravity waves reflected from a vortex sheet are amplified if the shear flow speed exceeds the horizontal phase speed of the incident gravity wave, i.e. if a critical level exists. The implication of these results is that under certain conditions gravity waves, rather than being absorbed, can extract energy from the mean flow. It is the purpose of the present work to indicate the different conditions that give rise to either critical-level behaviour (i.e. absorption), wave amplification or both.

As a model, we have chosen our basic flow to be a finite uniform shear layer.

This model has the advantage that some analytical results can be obtained. Concerning its stability properties, the following remarks may be in order. If the shear layer extends over a semi-infinite region, the flow is stable for all positive Richardson numbers (Chandrasekhar 1961). The stability of the finite shear layer considered here is not investigated, but we suspect that it may be unstable for sufficiently small Richardson numbers (see the review by Drazin & Howard 1966 for the stability of similar, but in important respects different, shear layers). Wave amplification also occurs for sufficiently small Richardson numbers but it is important to realize that, although wave amplification and instability are both related to the presence of streaming, they are in general quite distinct because they involve the excitation of different hydrodynamic modes. Hence the criteria for their occurrence are different (see, for example, Miles 1957; Fejer & Miles 1963; Fejer 1964; McKenzie 1970 for a discussion of the criteria for stability and wave amplification for vortex and current sheets).

In the next section we calculate the reflexion and transmission coefficients for a gravity wave incident upon a shear layer. In the last section we use the invariance of the wave-action flux to establish some general results. In the interesting situation in which the ultimate mean flow speed U_0 (on the far side of the shear layer) exceeds the phase speed c of the incident wave relative to the near side of the shear layer, so that a critical level exists, we find the following.

(i) There is a discontinuity in the net flux of wave action across the critical level for all positive Richardson numbers. For $Ri > \frac{1}{4}$, this discontinuity represents absorption of wave energy into the shear layer, whereas for $Ri \leqslant \frac{1}{4}$ it may be thought of as representing an energy flux 'defect' associated with the layer.

(ii) Wave amplification (or over-reflexion) occurs if the Richardson number is less than a certain critical value Ri_c, which for the case in hand turns out to be approximately 0·1129.

Thus if $Ri < R_c$ and $U_0 > c$, part of the incident wave's total energy flux is 'locked' inside the layer as an energy flux 'defect' and at the same time the reflected wave is amplified, thereby extracting energy from the streaming motion.

In the limiting case when the vertical wavelength (normalized by the thickness of the shear layer) is large and the associated Richardson number is small, we find that the reflexion coefficient for the total energy flux (or wave-action flux) is approximated by the remarkably simple expression

$$|R/I|^2 = U_0^2/(2c-U_0)^2.$$

This formula reveals not only the strong asymmetry of the properties of reflexion due to streaming but also the wave amplification that arises when $U_0 > c$ and the resonance that occurs at $U_0 = 2c$ because for this flow speed the incident wave stimulates a natural mode of the shear layer. It has been shown (McKenzie 1972) that wave amplification is due to the coupling of positive- and negative-energy waves on either side of the shear layer. A negative-energy wave is one that appears to carry a deficiency of energy in the laboratory frame of reference (Sturrock 1962), and this occurs for the transmitted wave if $U_0 > c$. It is pointed out that wave amplification could give rise to instability if the boundary conditions were altered in such a way that the system behaves like an 'amplifier'.

2. Reflexion and transmission of gravity waves by a uniform wind shear

For wave perturbations varying as $\exp i(\omega t - k_x x)$ in an incompressible stratified fluid where the density varies as $\exp(-\beta z)$ and which moves with mean velocity $\mathbf{U} = (U(z), 0, 0)$, the linearized equations of motion and continuity, under the Boussinesq approximation, can be combined to yield (Jones 1968)

$$\frac{d^2\phi}{dz^2} + \left\{\frac{N^2 k_x^2}{(\omega - k_x U^2)^2} - k_x^2 - \tfrac{1}{4}\beta^2 - \frac{\beta k_x}{(\omega - k_x U)}\frac{dU}{dz} + \frac{k_x}{(\omega - k_x U)}\frac{d^2U}{dz^2}\right\}\phi = 0. \quad (1)$$

Here ϕ is related to the vertical component of the perturbation velocity w through $w = \phi \exp(\frac{1}{2}\beta z)$, N is the Brunt–Väisälä frequency, given by $N^2 = \beta g$, and g the downward gravitational acceleration.

We consider the problem of a gravity wave incident upon a shear layer. The basic flow is specified by

$$U = \left\{\begin{array}{ll} 0, \quad z \leqslant 0 \quad (\text{region I}), \\ U'z, \quad U' = U_0/L, \quad 0 \leqslant z \leqslant L \quad (\text{region II}), \\ U_0, \quad z \geqslant L \quad (\text{region III}). \end{array}\right\} \quad (2)$$

A gravity wave from region I incident upon the wind shear layer (region II) gives rise to a reflected wave in region I, a transmitted wave in region III and two waves, one upgoing and the other downgoing, in region II. In region I the solution takes the form

$$\phi = I \exp(ik_{z1} z) + R \exp(-ik_{z1} z), \quad (3)$$

where I is the amplitude of the incident wave and R that of the reflected wave. The vertical wavenumber k_{z1} is given by

$$k_{z1}^2 = k_x^2(N^2/\omega^2 - 1) - \tfrac{1}{4}\beta^2. \quad (4)$$

If we take k_{z1} as the positive root of this equation, the choice of the signs in (3) ensures that the incident wave transports wave energy upwards (towards the shear layer) and the reflected wave carries wave energy downwards (away from the shear layer) since downward (upward) phase propagation of a gravity wave corresponds to upward (downward) wave energy propagation (see § 3). In region II equation (1) takes the form of Whittaker's equation, so that the solution may be written as

$$\phi = A_1 W_1(z) + A_2 W_2(z), \quad (5)$$

where A_1 and A_2 are amplitude constants and W_1 and W_2 are Whittaker functions (Jones 1968):

$$W_1(z) = M_{j,m}(\gamma(z_c - z)), \quad W_2(z) = M_{j,-m}(\gamma(z_c - z)), \quad (6)$$

in which

$$\left.\begin{array}{l} \gamma = 2k_x(1 + \beta^2/4k_x^2)^{\frac{1}{2}}, \quad j = \beta/\gamma, \quad z_c = c/U', \\ c = \omega/k_x, \quad m^2 = \tfrac{1}{4} - Ri, \quad Ri = N^2/U'^2. \end{array}\right\} \quad (7)$$

Here Ri is the Richardson number characterizing the layer, and the height z_c, usually called the critical level, where the horizontal phase speed c matches the flow speed U, corresponds to a regular singular point of (1).

In region III the solution takes the form

$$\phi = T\exp(ik_{z3}z), \tag{8}$$

where T is the amplitude of the transmitted wave and k_{z3} is given by

$$k_{z3}^2 = k_x^2\{N^2/(\omega - k_x U_0)^2 - 1\} - \tfrac{1}{4}\beta^2. \tag{9}$$

Again the choice of sign of k_{z3} ensures that the transmitted wave transports wave energy upwards (i.e. $k_{z3}(c - U_0) > 0$).

To simplify the algebra in the discussion given in the next section, we shall make use of the 'low frequency' approximation, i.e. $\omega \ll N$, so that the dispersion equations appropriate to regions I and III may be approximated by

$$k_{z1}^2 = \frac{N^2}{\omega^2}k_x^2, \quad k_{z3}^2 = \frac{N^2k_x^2}{(\omega - k_x U_0)^2}. \tag{10}$$

The amplitudes of the reflected and transmitted waves are determined by applying the boundary conditions at $z = 0$ and $z = L$. The boundary conditions, namely the continuity of the vertical component of velocity and continuity of pressure, are equivalent to

$$[\phi] = \left[\frac{d\phi}{dz} - \frac{\phi}{U-c}\frac{dU}{dz}\right] = 0 \quad \text{at} \quad z = 0, L, \tag{11}$$

where square brackets indicate the jump in the quantity within.

Some straightforward algebra then yields the following for the reflexion and transmission coefficients:

$$R/I = -(\delta_1 d_{i1} + d_{i2})/(\delta_1 d_{r1} + d_{r2}), \tag{12}$$

$$\frac{T\exp(ik_{z3}L)}{I+R} = \frac{\delta_1 W_1(L) + W_2(L)}{\delta_1 W_1(0) + W_2(0)} \tag{13}$$

for $U_0 < c$ and

$$R/I = -(\delta_2 d_{i1} + d_{i2})/(\delta_2 d_{r1} + d_{r2}), \tag{14}$$

$$\frac{T\exp(ik_{z3}L)}{I+R} = -ie^{-\pi mi}\frac{\delta_2 W_1(-L) + W_2(-L)\,e^{2\pi mi}}{\delta_2 W_1(0) + W_2(0)} \tag{15}$$

for $U_0 > c$, where

$$\left.\begin{aligned} d_{in} &= \gamma W_n'(0) + (ik_{z1} - z_c^{-1})\,W_n(0), \quad d_{rn} = \gamma W_n'(0) - (ik_{z1} + z_c^{-1})\,W_n(0),\\ \delta_1 &= c_{t2}/c_{t1}, \quad \delta_2 = -e^{2\pi mi}b_{t2}/b_{t1},\\ c_{tn} &= \gamma W_n'(L) + \{ik_{z3} - (L - z_c)^{-1}\}\,W_n(L),\\ b_{tn} &= \gamma W_n'(-L) - \{ik_{z3} + (L - z_c)^{-1}\}\,W_n(-L), \end{aligned}\right\} n = 1, 2. \tag{16}$$

Here we have used the notation

$$W_{1,2}(L) = M_{j,\pm m}(\gamma(z_c - L)), \quad W_{1,2}(0) = M_{j,\pm m}(\gamma z_c), \quad W_{1,2}(-L) = M_{j,\pm m}(\gamma(L - z_c)). \tag{17}$$

In matching the solutions across the critical level, we have used the results of Baldwin & Roberts (1970), which apply for all values of the Richardson number (see also Miles 1961; Booker & Bretherton 1967).

The dispersion equation for the system as a whole may be obtained by putting $I = 0$ in the boundary conditions and equating to zero the determinant of the coefficients of the amplitude factors. This leads to

$$\delta_1 d_{r1} + d_{r2} = 0 \quad \text{for} \quad U_0 < c \tag{18}$$

and

$$\delta_2 d_{r1} + d_{r2} = 0 \quad \text{for} \quad U_0 \geqslant c. \tag{19}$$

Thus the zeros of the denominators in (12) or (14) correspond to the normal modes (stable or unstable) of the system. Therefore it may be expected that, if the horizontal phase velocity of an incident wave matches the corresponding phase speed of the system, we shall have resonance (i.e. $|R/I| \rightarrow \infty$). More generally, it will be shown, in the next section, that incident waves whose vertical wavelengths are in some sense large compared with the thickness of the shear layer can be amplified ($|R/I| > 1$) on reflexion from the layer provided that the Richardson number is sufficiently small and $U_0 > c$.

3. Discussion of wave amplification and critical-level behaviour

We now discuss how a gravity wave incident upon a shear layer exhibits different reflectivity and transmissivity properties depending on the Richardson number, the wavelength, and the length scale of the wind shear. Booker & Bretherton (1967) have shown that, if the Richardson number is of order unity or larger, a critical layer forms around the level where the horizontal phase speed of the wave matches the flow speed. Since the reflexion coefficient is very small and the wave is heavily attenuated in passing through the critical layer, the implication is that the shear layer absorbs the energy of the incident wave. However, if $U_0 < c$, no critical level exists and the wave is transmitted with constant total energy flux (or, equivalently, constant wave-action flux, defined in (20) below).

On the other hand an analysis (McKenzie 1972) of gravity waves incident upon a vortex sheet (i.e. a discontinuous change in wind speed) exhibited wave amplification provided that the jump in the flow speed exceeded the horizontal phase speed of the incident gravity wave. In this case the wave extracts energy from the wind shear. As we shall see, that analysis provides the asymptotic form for the reflexion coefficient for a finite shear when the wavelength is very much greater than the length scale L of the shear.

We shall establish some general results by using the invariant

$$A = \mathrm{Re}\{(-i\bar{\phi}\, d\phi/dz)/2k_x^2\}, \tag{20}$$

where Re denotes the real part and the bar the complex conjugate. The fact that A is independent of z, except at the critical level (where it is discontinuous), follows from the differential equation (1) and the boundary conditions (11). A is called the flux of wave action (Bretherton & Garrett 1968; Hayes 1970) and its invariance is closely linked to the invariance both of the vertical component of the total energy flux and of the vertical flux of the horizontal component of momentum, or Reynolds stress (Eliassen & Palm 1960).

To clarify the discussion in the next section on the results obtained below using the wave-action flux, we mention here the relationships between wave-action flux, wave energy flux and total energy flux. If p is the perturbation pressure, w is the vertical component of the perturbation velocity and angular brackets denote a time average, the wave-action flux is $\langle pw/k(c-U)\rangle$ and the total energy flux is $\langle cpw/(c-U)\rangle$ while the wave energy flux is merely $\langle pw\rangle$. For the purpose of defining the concept of positive- and negative-energy waves (see below) we find it convenient to think in terms of the wave energy density and wave momentum density. To find the wave energy density ϵ and the wave momentum density $\mathbf{M}$, in a reference frame relative to which the fluid moves uniformly along the x axis with speed U, we can use the rules for the Galilean transformation of the wave energy–momentum tensor (Sturrock 1962). These yield $M_x = M'_x$ and $\epsilon = \epsilon' + UM'_x$, where ϵ' and M'_x are respectively the wave energy density and the x component of the wave momentum density in the rest frame of the fluid. From the relations $M_x = \epsilon' k'_x/\omega'$ and $\omega' = \omega - k_x U$ we obtain, after using $k_x = k'_x$, $\epsilon = \epsilon'\omega/\omega' = \epsilon c/(c-U)$. Since ϵ', $\omega > 0$, we see that the wave energy density ϵ in the laboratory frame is *negative* if $\omega' < 0$. Thus $\epsilon \gtrless 0$ according as $U \lessgtr c$. On the other hand the x component $\epsilon' k_x/\omega'$ of the wave momentum density (or wave pseudo-momentum; see McIntyre 1975) is invariant. The wave action is given by ϵ'/ω' and the vertical component of the total energy flux is

$$\epsilon\, \partial\omega/\partial k_z = (\epsilon'\omega/\omega')\, \partial\omega'/\partial k_z.$$

Now if $U_0 < c$ there is no critical level in the shear layer, and the invariance of A over the whole domain gives

$$k_{z1}\{|I|^2 - |R|^2\} = k_{z3}|T|^2, \tag{21}$$

which may also be interpreted as expressing equality of total energy flux into and out of the layer. In accordance with the rules for the choice of sign both k_{z1} and k_{z3} are positive, thus $|R|^2 < |I|^2$ and wave amplification is impossible for all Richardson numbers when $U_0 < c$.

If $U_0 > c$, there is a critical level within the layer; we make use of the following approximate solutions valid near $z = z_c$:

(*a*) $Ri > \frac{1}{4}$,

$$\phi = \begin{cases} A_1\{\gamma(z_c - z)\}^{\frac{1}{2}+i\mu} + A_2\{\gamma(z_c - z)\}^{\frac{1}{2}-i\mu}, & z < z_c, \\ A_1^*\{\gamma(z - z_c)\}^{\frac{1}{2}+i\mu} + A_2^*\{\gamma(z - z_c)\}^{\frac{1}{2}-i\mu}, & z > z_c, \end{cases} \tag{22}$$

where $$A_1^* = -iA_1 e^{\pi\mu}, \quad A_2^* = -ie^{-\pi\mu}, \quad \mu = (Ri - \tfrac{1}{4})^{\frac{1}{2}}; \tag{23}$$

(*b*) $Ri < \frac{1}{4}$,

$$\phi = \begin{cases} A_1\{\gamma(z_c - z)\}^{\frac{1}{2}+m} + A_2\{\gamma(z_c - z)\}^{\frac{1}{2}-m}, & z < z_c, \\ A_1^*\{\gamma(z - z_c)\}^{\frac{1}{2}+m} + A_2^*\{\gamma(z - z_c)\}^{\frac{1}{2}-m}, & z > z_c, \end{cases} \tag{24}$$

where $$A_1^* = -iA_1 e^{-im\pi}, \quad A_2^* = -iA_2 e^{im\pi}. \tag{25}$$

The matching of the solutions across the critical layer (Baldwin & Roberts 1970) is essentially the same (since $m = i\mu$) in both cases but is written in this form to illustrate the different natures of the solutions in the shear layer for values of the Richardson number below and above $\frac{1}{4}$.

Evaluating A below the critical level gives

$$k_{z1}\{|I|^2-|R|^2\} = Q_1, \tag{26}$$

where
$$Q_1 = \begin{cases} -\gamma\mu(|A_1|^2-|A_2|^2), & Ri > \frac{1}{4}, \\ -\gamma m\,\mathrm{Im}\,(A_2\bar{A}_1), & Ri < \frac{1}{4}. \end{cases} \tag{27}$$

Here Im denotes the imaginary part. Similarly we evaluate A above the critical level:

$$k_{z3}|T|^2 = Q_2, \tag{28}$$

where
$$Q_2 = \begin{cases} \gamma\mu\{|A_1|^2\exp(2\pi\mu)-|A_2|^2\exp(-2\pi\mu)\}, & Ri > \frac{1}{4}, \\ \gamma m\,\mathrm{Im}\,(A_2^* A_1^*), & Ri < \frac{1}{4}. \end{cases} \tag{29}$$

Combining (26) and (28) gives the following form for the energy equation when there is a critical level in the shear layer:

$$k_{z1}|I|^2 = k_{z1}|R|^2+k_{z3}|T|^2+Q_1-Q_2. \tag{30}$$

The term on the left-hand side of (30) represents the total energy flux into the shear layer whereas the first two terms on the right-hand side represent the total energy flux out of the layer. (Note that $k_{z3} < 0$ here.) In view of the different natures of the solutions in the layer for $Ri \gtrless \frac{1}{4}$ [see (22) to (25)] it would perhaps be misleading to interpret Q_1-Q_2, when $Ri \leqslant \frac{1}{4}$, as representing the flux of energy absorbed into the layer. In this case Q_1-Q_2 may be thought of as representing an energy flux 'defect' associated with the layer.

When $U_0 > c$, $k_{z1} > 0$ and $k_{z3} < 0$. It follows immediately from (27), (29) and (30) that there can be no wave amplification if $R_i > \frac{1}{4}$. On the other hand, when $Ri < \frac{1}{4}$, wave amplification requires $Q_1 < 0$. This result also follows directly from the reflexion coefficient (14).

If we note that

$$\mathrm{Im}\,(A_2\bar{A}_1) = (A_2\bar{A}_1-\bar{A}_2A_1)/2i = |A_2|^2(\bar{\delta}_2-\delta_2)/2i,$$

and use expression (16) for δ_2, we find that

$$Q_1 = 2\gamma m|A_2|^2(a_1^2+b_1^2)^{-1}\{-2\gamma k_{z3}m\cos 2\pi m+(a_1a_2+b_1b_2)\sin 2\pi m\}, \tag{31}$$

where
$$\left.\begin{aligned} a_{1,2} = \gamma\{M'_{j,\pm m}(x)-M_{j,\pm m}(x)/x\},\quad b_{1,2} = k_{z3}M_{j,\pm m}(x),\\ x = \gamma(L-z_c) = \gamma L(1-c/U). \end{aligned}\right\} \tag{32}$$

By noting that k_{z3} can be written as

$$k_{z3} = (\tfrac{1}{4}-m^2)^{\frac{1}{2}}/L(-1+c/U_0),$$

we see that
$$k_{z3}/2k_x = (\tfrac{1}{4}-m^2)^{\frac{1}{2}}/x = N/2\omega',$$

where ω' is the Doppler-shifted frequency. Since $\omega' \ll N$, then $x \ll 1$. Thus in (32) it is legitimate to use the following asymptotic forms of the Whittaker functions valid for small argument:

$$M_{j,\pm m}(x) \simeq x^{\frac{1}{2}\pm m}, \quad x \ll 1. \tag{33}$$

It follows that

$$Q_1 \simeq |A_2|^2(2\gamma)^2mk_{z3}(a_1^2+b_1^2)^{-1}\{-m\cos 2\pi m-(\tfrac{1}{4}-m^2)^{\frac{1}{2}}\sin 2\pi m\}. \tag{34}$$

Since $k_{z3} < 0$, the condition for wave amplification ($Q_1 < 0$) is fulfilled if

$$-\cot 2\pi m > (-1 + 1/4m^2)^{\frac{1}{2}}, \tag{35}$$

which yields

$$Ri < Ri_c \quad (= 0\cdot 1129). \tag{36}$$

Thus wave amplification occurs if the ultimate shear flow speed is greater than the phase speed of the incident wave provided that the Richardson number of the shear layer is less than $Ri_c = 0\cdot 1129$.

On the other hand we note that critical-level behaviour occurs not only for $Ri > \frac{1}{4}$ but also for $Ri < \frac{1}{4}$. In the latter case the amount of wave energy absorbed or locked into the layer [see remarks below (30)] is proportional to $Q_1 - Q_2$, which is given by

$$Q_1 - Q_2 \simeq |A_2|^2 (2\gamma)^2 m(-k_{z3})(a_1^2 + b_1^2)^{-1} \{m(1 + \cos 2\pi m) + (\tfrac{1}{4} - m^2)^{\frac{1}{2}} \sin 2\pi m\} \tag{37}$$

when use is made of (33) for the Whittaker functions. Thus we see that $Q_1 - Q_2$ is positive and tends to zero only when $R_i \to 0$ (i.e. $m \to \frac{1}{2}$).

We shall now obtain the asymptotic forms of the reflexion coefficient in the limits (i) $\gamma L \ll 1$ or $Ri \ll 1$ and (ii) $Ri \gg 1$.

Case (i). $\gamma L \ll 1$

When the normalized wavelength is large (i.e. $\gamma L \ll 1$), the Whittaker functions are approximated by (33) and (12)–(17) yield to first order

$$\frac{R}{I} = \begin{cases} \dfrac{k_{z1} z_c^2\{-1 + ik_{z3}(z_c - L)\} + k_{z3}(z_c - L)^2 (1 - ik_{z1} z_c)}{k_{z1} z_c^2\{-1 + ik_{z3}(z_c - L)\} - k_{z3}(z_c - L)^2 (1 + ik_{z1} z_c)}, & U_0 < c, \\[2ex] \dfrac{ik_{z1} z_c^2\{\frac{1}{2} + m + ik_{z3}(L - z_c)\} - (L - z_c)^{2m}\{m - \frac{1}{2} - ik_{z3}(L - z_c)\} e^{-2\pi m i}}{ik_{z1} z_c^2\{\frac{1}{2} + m + ik_{z3}(L - z_c)\} + (L - z_c)^{2m}\{m - \frac{1}{2} - ik_{z3}(L - z_c)\} e^{-2\pi m i}}, & U_0 \geqslant c, \end{cases} \tag{38}$$

and

$$\frac{T \exp(ik_{z3} L)}{I + R} = \begin{cases} \dfrac{2m(1 - U_0/c)^{\frac{1}{2}+m}}{2m + ik_{z3}(z_c - L)\{(1 - U_0/c)^{2m} - 1\}}, & U_0 < c, \\[2ex] \dfrac{-2im\, e^{-i\pi m}(U_0/c - 1)^{\frac{1}{2}+m}}{m + \frac{1}{2} + ik_{z3}(L - z_c) + \{m - \frac{1}{2} - ik_{z3}(L - z_c)\}(U_0/c - 1)^{2m} e^{-2\pi m i}}, & U_0 \geqslant c. \end{cases} \tag{39}$$

These expressions simplify when we take the limit $m \to \frac{1}{2}$ ($Ri \to 0$) to

$$\left.\begin{aligned} R/I &= (k_{z3} - k_{z1})/(k_{z3} + k_{z1}) \quad (40) \\ T/(I + R) &= 1 - U_0/c \quad (41) \end{aligned}\right\} \text{ for all } U.$$

These agree with the reflexion and transmission coefficients for acoustic–gravity waves incident upon a sharp discontinuity in velocity (McKenzie 1972) when we use the approximation $\omega \ll N$. In accordance with the rules for the choice of sign of k_{z1} and k_{z3} we obtain the following expressions for the reflexion and transmission coefficients for the total energy flux expressed as a function of U_0 and c:

$$|R/I|^2 = U_0^2/(2c - U_0)^2, \tag{42}$$

$$\tau^2 = 4(c - U_0)^3/c(2c - U_0)^2. \tag{43}$$

Here τ^2 is the ratio of the transmitted energy flux to the incident flux and care has been taken to define the energy density of the transmitted wave in a moving medium [McKenzie 1972; see remarks above (21)] with the result that

$$\tau^2 = \frac{1}{(1-U_0/c)^2}\left|\frac{T}{I}\right|^2 \frac{k_{z1}}{k_{z3}}. \tag{44}$$

These equations display the features of wave amplification, resonance and the strong asymmetry of the reflexion coefficient about $U_0 = 0$. We see that the wave amplification occurs when the fluid in region III moves faster than the horizontal phase speed of the incident wave. This phenomenon is due to the interaction between positive- and negative-energy waves (McKenzie 1972), in the sense explained above (21), in the presence of streaming. Here the incident wave carries positive energy towards the shear layer while the transmitted wave carries away negative energy, that is to say, the transmitted wave appears to carry a deficiency of energy in the laboratory frame as a result of the 'supersonic' motion. The resonance ($|R/I|^2 \to \infty$) at $U_0 = 2c$ arises because the phase speed of the incident wave matches the phase speed of a natural mode of the shear layer. This can be seen by considering the dispersion equations (18) and (19) in the limit $\gamma L \ll 1$. This gives

$$k_{z1} + k_{z3} = 0 \tag{45}$$

in both cases, which, on using the expressions for k_{z1} and k_{z3}, shows that the phase velocity of a 'ripple' on a velocity 'discontinuity' is one half the jump in the fluid speed across the discontinuity, i.e. $\omega/k_x = \frac{1}{2}U_0$. We also note that the reflexion coefficient tends to unity for flow speeds well in excess of the horizontal phase speed. In fact, however, total reflexion takes place at flow speeds above that required to Doppler shift the frequency up to the Brunt–Väisälä frequency. That this feature does not emerge from (42) is a result of our 'low frequency' approximation.

Although wave amplification is not a result of instability, it is intuitively clear that it could be a source of instability. For example, consider altering the boundary conditions in region I by introducing a solid boundary at some depth $z = -d$. From the analysis above, a wave reflected from the shear layer can be amplified if $U_0 > c$. This reflected wave will be perfectly reflected from the solid boundary and on a further reflexion from the shear layer will again be amplified and so on. Thus the wave picks up more and more energy and momentum from the shear layer as it bounces between it and the solid boundary. In this way the system behaves like an amplifier with positive feedback. Therefore it seems clear that wave amplification can lead to instability since such an alteration in the boundary conditions has provided a mechanism for transferring the available energy in the streaming motion into hydrodynamic modes: in this case gravity waves.

Case (ii). $Ri \gg 1$

Let us now consider reflexion from and transmission through a shear layer characterized by a large Richardson number and hence associated with a small normalized wavelength (i.e. $k_{z1}L \gg 1$). In this case we expect the reflexion coefficient to be small since the 'geometric optics' (or WKBJ) approximation should be valid, and at the same time the transmission of the wave will be strongly affected by whether or not a critical level is present within the layer.

The approximate solutions (22) are appropriate for this case of large Richardson number and by writing these functions in the form

$$(z_c - z)^{\frac{1}{2} \pm iRi^{\frac{1}{2}}} = (z_c - z)^{\frac{1}{2}} \exp\{\pm iRi^{\frac{1}{2}} \log(z_c - z)\} \tag{46}$$

we see that the wave perturbations behave as $\exp i\{\omega t - k_x x \pm Ri^{\frac{1}{2}} \log(z_c - z)\}$, which shows that the local vertical wavenumber k_{z2}, say, is given by

$$k_{z2} = \mp Ri^{\frac{1}{2}}/(z_c - z). \tag{47}$$

In interpreting (22) as representing the sum of upgoing and downgoing waves, we see that the wave amplitudes are altered by the factors $-i \exp \pm Ri^{\frac{1}{2}}$. Thus for $Ri \gg 1$, $|\delta_2|$ $(= |A_1/A_2|)$ is either very small or very large, in which case the reflexion coefficient (14) can be approximated by

$$\frac{R}{I} = -\frac{\gamma W_n'(0)/W_n(0) - ik_{z1} + z_c^{-1}}{\gamma W_n'(0)/W_n(0) + ik_{z1} + z_c^{-1}}, \tag{48}$$

where $n = 1$ or 2 and we must choose the Whittaker function W_n that represents a wave carrying (wave) energy upwards, towards the critical level. From (47) above and (22), we see that $A_2 W_2$ is the upward-propagating wave which is absorbed in the critical layer. Therefore, from (22), we have

$$\gamma W_n'(0)/W_n(0) = -(\tfrac{1}{2} - iRi^{\frac{1}{2}})/z_c. \tag{49}$$

Substituting (49) along with $k_{z1} z_c = Ri^{\frac{1}{2}}$ into (48), we obtain

$$R/I = i/4Ri^{\frac{1}{2}}. \tag{50}$$

Thus, in accordance with our expectations, (50) shows that the reflected energy flux is small, being of order $1/16Ri$. It is worth noting that this reflected energy flux arises from partial reflexions of the upgoing wave before it reaches the critical level. These are dominated by the reflexion from the shear discontinuity at $z = 0$.

In a similar way we find that the transmission coefficient given by (13) and (15) approximates to

$$\frac{T}{I} \exp(ik_{z3} L) = \begin{cases} (1 - U_0/c)^{\frac{1}{2} + iRi^{\frac{1}{2}}}, & U_0 < c, \\ i \exp(-\pi Ri^{\frac{1}{2}})(U_0/c - 1)^{\frac{1}{2} + Ri^{\frac{1}{2}}}, & U_0 \geqslant c. \end{cases} \tag{51}$$

It is instructive to use (51) to write down the ratio of the transmitted and incident energy fluxes. The vertical component of the total energy flux of an acoustic-gravity wave in a medium moving along the x axis with speed U is given by (McKenzie 1972)

$$\begin{aligned} J_z &= \frac{\rho w^2 \omega k_z (\omega'^2 - N^2)}{(k_z^2 + \beta^{-1})\,\omega'^2} \\ &\simeq \rho w^2 N^2 \omega / (-k_z)\,\omega'^2, \quad \omega' \ll N, \end{aligned} \tag{52}$$

where $\omega' - \omega - k_x U$ and w is the vertical component of the perturbation velocity. Substituting (51) into (52), we find that the ratio of the transmitted and incident total energy fluxes (or equivalently, wave-action fluxes) is given by

$$J_{z\,\mathrm{trans}}/J_{z\,\mathrm{inc}} = \begin{cases} 1, & U_0 < c, \\ \exp(-2\pi Ri)^{\frac{1}{2}}, & U_0 > c. \end{cases} \tag{53}$$

When $U_0 < c$, there is no critical level and we have complete transmission of the energy, although the amplitude of the transmitted wave is not equal to that of the incident wave. It is interesting to note that the reason for this lies in the fact that the energy density of a wave is not invariant under Galilian transformations of the reference frame whereas the momentum flux density is invariant under such transformations (Sturrock 1962). On these grounds it can be deduced that, for gravity waves in a shear flow, the upward flux of horizontal momentum is conserved except at a critical level, where there is a discontinuous jump (Eliassen & Palm 1960; Booker & Bretherton 1967), whereas the energy density is not.

When $U_0 > c$, equation (53) serves to show that the transmission of total energy flux is practically zero, because of strong absorption into the shear layer.

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